1. Suppose that $X_1, X_2, \ldots$ are independent and identically distributed $\mathrm{N}(0,1)$ random variables. Let

$$Y_i = \begin{cases} X_i - 1 & \text{if } X_i \leq 0, \\ X_i & \text{if } X_i > 0, \end{cases} \quad i = 1, 2, \ldots.$$

   (a) Find the mean and variance of Y_1.

   (b) Find constants α_n and β_n, depending on n, such that

$$\alpha_n \sum_{i=1}^{n} Y_i - \beta_n \text{ converges in distribution to } Z \text{ as } n \to \infty,$$

   where Z has a standard normal distribution.

2. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with one of two probability density functions $f(x|\theta), \theta = 0, 1$. If $\theta = 0$, then

$$f(x|\theta) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

while if $\theta = 1$, then

$$f(x|\theta) = \begin{cases} \dfrac{1}{2\sqrt{x}} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the maximum likelihood estimator $\hat{\theta}_n$ of θ.

   (b) Show that $\lim_{n\to\infty} P_{\theta=0}(\hat{\theta}_n = 0) = 1$.

3. Let $X_1, X_2, \ldots, X_n$ $(n \geq 2)$ be independent and identically distributed random variables having uniform distribution over $\{1, 2, \ldots, \theta\}$, where $\theta \in \{1, 2, \ldots\}$.

   (a) Let $X_{(n)} = \max(X_1, \ldots, X_n)$. Show that $X_{(n)}$ is sufficient for θ.

   (b) We wish to test the hypothesis $\mathrm{H}_0 : \theta \leq \theta_0$ against $\mathrm{H}_1 : \theta > \theta_0$, where θ_0 is a known positive integer. For $0 < \alpha < 1$, consider the test

$$T(X_{(n)}) = \begin{cases} 1 & \text{if } X_{(n)} > \theta_0, \\ \alpha & \text{otherwise.} \end{cases}$$

   Show that T is a uniformly most powerful test of size α for testing H_0 against H_1.

4. Consider an urn containing 10 red balls, 10 white balls, and 10 black balls. Balls are drawn at random with replacement one by one. Let T be the minimum number of draws required to get balls of three colours. Find the distribution of T.

5. Let $\{X_n : n \geq 1\}$ be a sequence of independent and identically distributed random variables, each having an exponential distribution with mean 1. Let $M_n = \max\{X_k : 1 \leq k \leq n\}$. Show that

$$\frac{M_n}{\log n} \xrightarrow{p} 1 \quad \text{as} \quad n \to \infty.$$

6. Consider the multiple linear regression model (with n subjects and p predictors) $\boldsymbol{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\boldsymbol{\epsilon} \sim \mathrm{N}_n(\mathbf{0}, \sigma^2\mathbf{I}_n)$.

   (a) Assume $\operatorname{rank}(\mathbf{X}) = p$. Obtain the uniformly minimum variance unbiased estimator of σ^2.

   (b) Suppose p is a fixed positive integer. Show that the estimator in (a) is a consistent estimator of σ^2.

   (c) Suppose now that both n and p vary such that $n - p \to \infty$. Show that the estimator in (a) is a consistent estimator of σ^2.

7. Consider a population $U = \{1, 2, 3\}$ of size 3. Let Y be a variable taking value Y_i on unit i, $i = 1, 2, 3$. The population mean is $\bar{Y} = (Y_1 + Y_2 + Y_3)/3$. A sample of size 2 is drawn from U by using simple random sampling *without* replacement. Let T be the sample mean. Consider the following estimator:

$$T^* = \begin{cases} \frac{1}{2}Y_1 + \frac{1}{2}Y_2 & \text{if units 1 and 2 are selected,} \\ \\ \frac{1}{2}Y_1 + \frac{2}{3}Y_3 & \text{if units 1 and 3 are selected,} \\ \\ \frac{1}{2}Y_2 + \frac{1}{3}Y_3 & \text{if units 2 and 3 are selected.} \end{cases}$$

   (a) Prove that T^* is an unbiased estimator of $\bar{Y}$.

   (b) Show that $\mathrm{Var}(T^*) < \mathrm{Var}(T)$ if $Y_3(3Y_2 - 3Y_1 - Y_3) > 0$.

8. Consider the Laplace distribution with probability density function $f(x|\theta)$ given by

$$f(x|\theta) = \frac{1}{2}\exp\left(-|x - \theta|\right), \; x \in \mathbb{R}, \; \theta \in \mathbb{R}.$$

   Prove that the family $\{f(x|\theta) : \theta \in \mathbb{R}\}$ is *not* a one-parameter exponential family.